Number of connected components of polynomial lemniscates

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The Setting

Polynomial Lemniscate

Let $p(z)$ be a monic complex polynomial. The *filled unit lemniscate* of p is defined by

$$\Lambda_p = \{z \in \mathbb{C} : |p(z)| < 1\}.$$
$$\partial\Lambda_p = \{z \in \mathbb{C} : |p(z)| = 1\}.$$

- The zero set of p, is always inside the lemniscate Λ_p.
- It is a *bounded, open* set with *rectifiable* boundary.
- Let $C(\Lambda_p) :=$ # Connected components of Λ_p.
- **Example-1:** Let $p(z) = z^n$. Then the *lemniscate* of z^n is

$$\Lambda_{z^n} = \{z \in \mathbb{C} : |z^n| < 1\} = \mathbb{D}.$$

Zeros of p on the circle of radius $\frac{1}{2}$

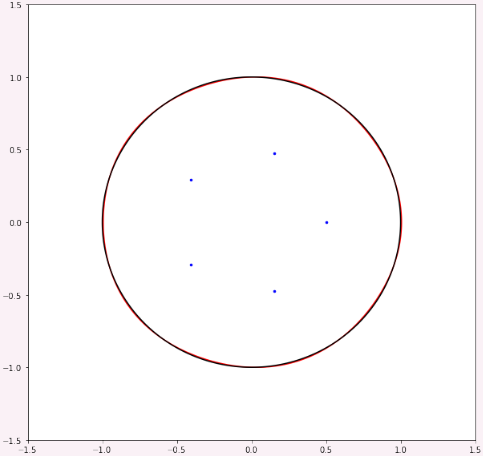


Figure: Lemniscate of $z^5 - \frac{1}{2}^5$.

Zeros of p on the circle of radius 1

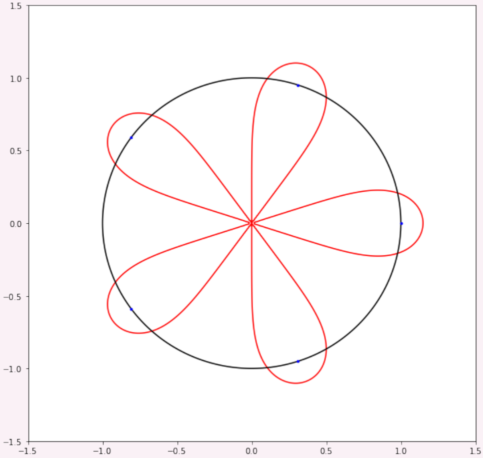


Figure: Lemniscate of $z^5 - 1^5$.

Zeros of p on the circle of radius 1.15

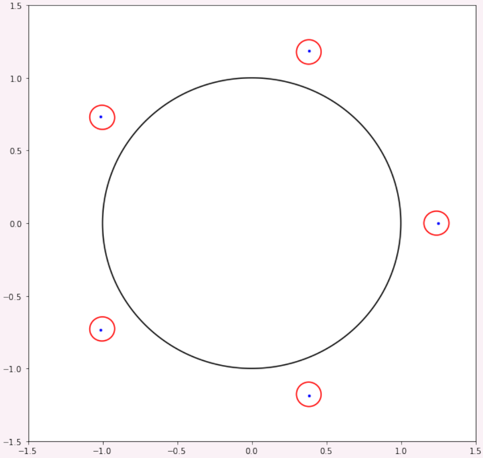


Figure: Lemniscate of $z^5 - 1.15^5$.

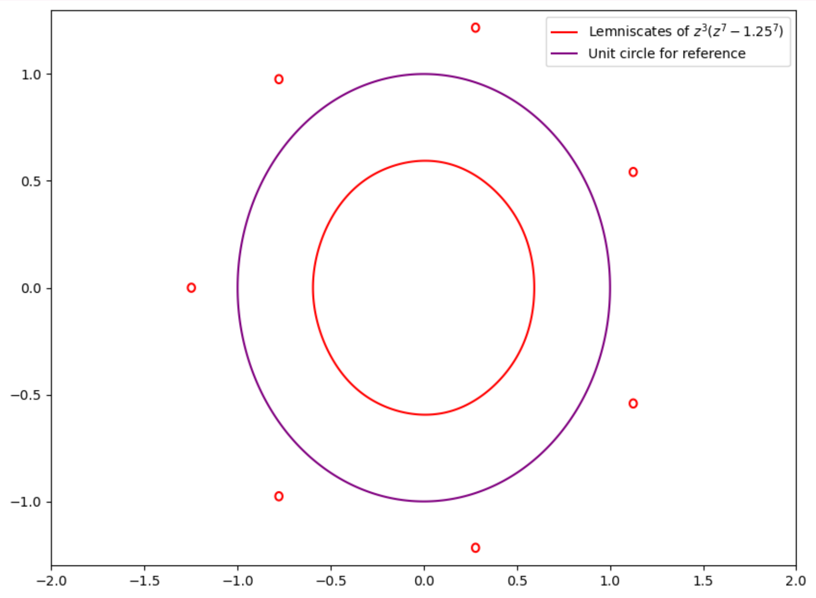
Can we get any number of components?

- The maximum modulus principle $\implies$ Each component of Λ_p contains a zero of p.

$$1 \leq C(\Lambda_p) \leq \deg(p).$$

- Without any restriction on the roots, given $k \leq n$, $\exists$ a polynomial p_n of degree n with k components.

$$z^{n-k+1}(z^{k-1} - a^{k-1})$$


Lemniscates of $z^3(z^7 - 1.25^7)$
Unit circle for reference
1.0
0.5
0.0
−0.5
−1.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

A Question by Erdös, Herzog, and Piranian

Questions:

- If we constrain all the zeroes of p to lie on a fixed compact set K, then how large is

$$\mathcal{C}_{\max}(K,n) := \max\{C(\Lambda_p) : \deg(p) = n, Z_p \subset K\}$$

$$\mathcal{C}_{\max}(\overline{\mathbb{D}},n) = n \quad \& \quad \mathcal{C}_{\max}(\{0\},n) = 1$$

- What **geometric features of K** determine the $\mathcal{C}_{\max}(K,n)$?

Special case of disk: $K = r\overline{\mathbb{D}}$

- Let $K_0 \subset K_1$, then

$$1 \leq \mathcal{C}_{\max}(K_0, n) \leq \mathcal{C}_{\max}(K_1, n) \leq n$$

Therefore, for $r \geq 1$ in $r\overline{\mathbb{D}}$

$$\mathcal{C}_{\max}(r\overline{\mathbb{D}}, n) = n$$

- Suppose, $r = \frac{1}{2}$, then for any

$$p(z) := \prod_{1}^{n}(z - a_i), \quad \textit{where} \quad |a_i| \leq \frac{1}{2}, \quad \forall i = 1, \cdots, n.$$

- If $z_0 \in \frac{1}{2}\mathbb{D}$ we have by the triangle inequality

$$|z_0 - a_i| < 1 \implies |p(z_0)| = \prod_{i} |z_0 - a_i| < 1.$$

- Therefore, $\frac{1}{2}\mathbb{D} \subset \Lambda_p \implies \mathcal{C}(\Lambda_p) = 1$
- Therefore, for $r \leq \frac{1}{2}$ in $r\overline{\mathbb{D}}$

$$\mathcal{C}_{\max}(r\overline{\mathbb{D}}, n) = 1$$

	$r \leq \frac{1}{2}$	$\frac{1}{2} < r < 1$	$r \geq 1$
$C_{\max}$	1	?	n

- Then it is natural to ask what is the maximum number of components if zeroes are in $r\overline{\mathbb{D}}$ when $r = \frac{1}{2} + \epsilon$ or $r = 1 - \epsilon$?

Lemniscates and Potentials

$$\Lambda_p = \{z \in \mathbb{C} : |p(z)| < 1\} = \{z \in \mathbb{C} : \log|p(z)| < 0\}$$
$$= \left\{z \in \mathbb{C} : \frac{1}{\deg(p)} \sum_{z_0 \in Z_p} \log|z - z_0| < 0\right\}$$

- Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials and let

$$Z_{p_n} := \frac{1}{n} \sum_{z_0 : p_n(z_0)=0} \delta_{z_0}$$

- Then $\exists$ a subsequence n_k and measure μ such that,

$$Z_{p_{n_k}} \xrightarrow{\text{weak}^*} \mu.$$

- Roughly: $\Lambda_{p_{n_k}} \xrightarrow{k\to\infty} \{\int_{\mathbb{D}} \log|z - w| d\mu(w) < 0\}.$

Potential Theory Preliminaries

Let $K \subset \mathbb{C}$ be a non-empty compact set, and μ a Borel probability measure on K.

- The logarithmic potential of μ is the function $U_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$U_\mu(z) = \int_K \log|z-w| d\mu(w), \; z \in \mathbb{C}.$$

Studying lemniscates $\sim$ Studying Sublevel set of potentials

Example: Let $\mu = (1-\epsilon)\delta_{-r} + \epsilon \mathcal{H}^1 | \Gamma_{AB}$, then

$$U_\mu(z) > 0$$

in a nbd of the arc Γ_{AB}, for ϵ small enough!

Equilibrium measure and *Capacity* of a compact set

- The *energy* associated with the measure μ is

$$I(\mu) = \int_K U_\mu(w) d\mu(w).$$

- The logarithmic capacity of K is

$$\log c(K) = \sup_{\mu \in \mathcal{P}(K)} I(\mu),$$

$\mathcal{P}(K) = \{\text{Borel probability measures on } K.\}$

- $\exists$ a unique probability measure called *equilibrium measure* $\nu = \nu_K$ on K, which satisfies

$$\log c(K) = I(\nu_K).$$

Random Setting

- Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables with law μ, supported in $r\overline{\mathbb{D}}$.
- Consider the **random polynomial**

$$P_n(z) := \prod_{i=1}^{n}(z - X_i).$$

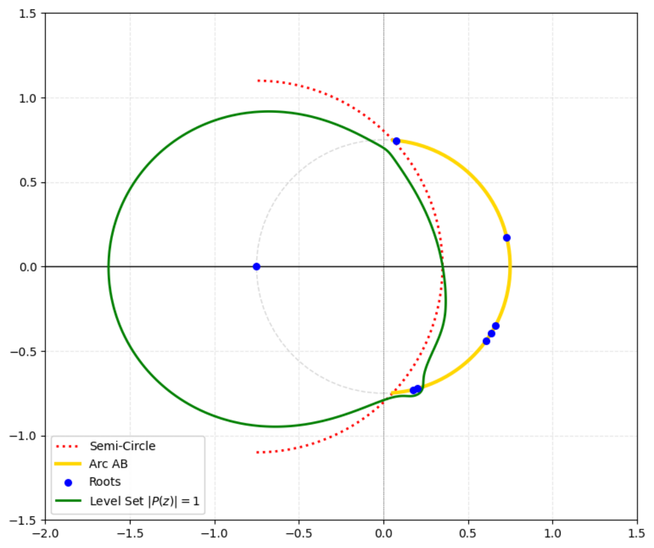
- The associated random lemniscate is

$$\Lambda_n(\omega) := \Lambda_{P_n} = \{z \in \mathbb{C} : |P_n(z)| < 1\}.$$
$$C(\Lambda_n)(\omega) := \textit{No. of components of the lemniscate.}$$

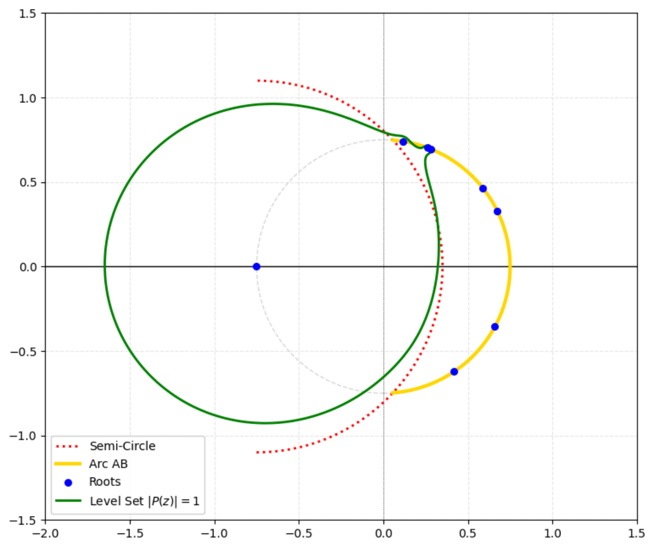
Goal: There is a positive constant C_r such that

$$\mathbb{E}[C(\Lambda_n)] \geq C_r n.$$

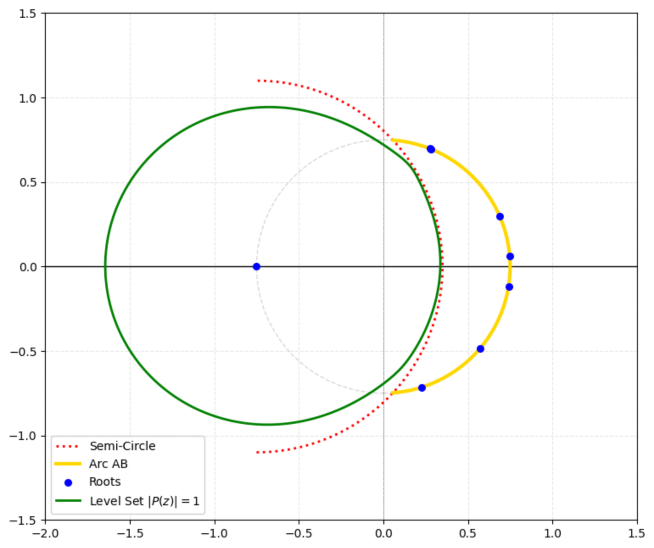
Simulations:



Simulations:



Simulations:



Lonely Component

- We say that a root z_j forms a *lonely component* if there exists a ball $\mathcal{B}$ containing z_j such that,

$$\begin{cases} z_k \notin \mathcal{B}, & \forall k \neq j \\ |P_n(z)| \geq 1, & \forall z \in \partial\mathcal{B}. \end{cases}$$

- Let $L_i = \{X_i \text{ forms an isolated component}\}$. Then it immediately follows that

$$C(\Lambda_n) \geq \sum_{i=1}^{n} \mathbb{1}_{L_i}$$

- Now taking the expectation

$$\mathbb{E}\left[C(\Lambda_n)\right] \geq \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{1}_{L_i}\right] \geq n\mathbb{E}\left[\mathbb{1}_{L_1}\right] \geq n\mathbb{P}\left(L_1\right) = C_r n.$$

Theorem (Ghosh & Ramachandran, JMAA 2024)

Capacity of K	Additional assumptions on K	$\mathcal{C}_{\max}(K,n)$ (for large n).
$c(K) < 1$	-	$< A_K n$ ($A_K < 1$)

Question: Can we get a lower bound for $\mathcal{C}_{\max}(K,n)$?

Theorem (Ghosh & Ramachandran, JMAA 2024)

Capacity of K	Additional assumptions on K	$\mathcal{C}_{\max}(K, n)$ (for large n).
$c(K) < 1$	-	$< A_K n$ ($A_K < 1$)
$c(K) > 1$	Connected	n

Question: What is $\mathcal{C}_{\max}(K, n)$ when K is the *cantor set* ?

Theorem (Ghosh & Ramachandran, JMAA 2024)

Capacity of K	Additional assumptions on K	$\mathcal{C}_{\max}(K,n)$ (for large n).
$c(K) < 1$	-	$< A_K n$ ($A_K < 1$)
$c(K) > 1$	Connected	n
$c(K) = 1$	Closure of Jordan domain with C^2 boundary	$\sim n$ (upto a subsequence)

Recent Breakthroughs in Polynomial Lemniscates

1. The Maximal Length Problem

(Erdős–Herzog–Piranian Conj.)

- **Question:** For a monic polynomial P of degree n, is the length of the level set $\partial\Lambda_p = \{z : |p(z)| = 1\}$ maximized by $p(z) = z^n - 1$?

Theorem (Tao, December 2025)

For all sufficiently large n, the conjecture holds. Specifically,

$$\text{Length}(\partial\Lambda_p) \leq 2\pi n + O(1).$$

2. The Minimal Area Problem

(Erdős;zeros in $\mathbb{D}$)

- **Question:** What is the minimal area of the sublevel set $\Lambda_p = \{z : |p(z)| < 1\}$ if all zeros of p lie in $\overline{\mathbb{D}}$?

Theorem (Krishnapur, Lundberg, Ramachandran, March 2025)

Let $A_n = \min_{p\in\mathcal{P}_n(\mathbb{D})} \text{Area}(\Lambda_p)$. Then,

$$\frac{c}{\log n} \leq A_n \leq \frac{C}{\log\log n}.$$

Thank you for your attention